Subject	The optimum position for a tidal power barrage in the Severn estuary				
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Installation	Severn Barrage	**Query Reference**	-		
Written	RCTR	**Checked**	FJMF	**Approved**	
Reference Documents	1) Baker, A.C. 1991 *Tidal Power* Peter Peregrinus Ltd. on behalf of the Institution of Electrical Engineers 2) Bondi, Sir Hermann *et al*. 1981 *Tidal Power from the Severn Estuary.* Dept. Energy, Energy Paper No.46. HMSO. 3) Lamb, Sir *Horace* 1932 *Hydrodynamics* 6th Ed. Cambridge University Press. 4) Lighthill, Sir *James* 1978 *Waves in Fluids* Cambridge University Press. 5) Rainey, R.C.T. 2001 *The Pelamis wave energy converter: it may be jolly good in practice, but will it work in theory?* Proc. 16th Int'l. Workshop on Water Waves and Floating Bodies, Hiroshima, Japan. See www.iwwwfb.org 6) Taylor, Sir Geoffrey 1921 *Tides in the Bristol Channel* in *The Scientific Papers of Sir Geoffrey Ingram Taylor,* Vol 2. pp 185-192. Cambridge University Press 1960				
Attachments	MATHCAD spreadsheet "Calculation of Severn Barrage Powers"				

Summary

G.I.Taylor's approximate analytical solution for the tidal flow in the Severn estuary is extended to find the optimum location for a tidal power barrage, from the power point of view. It appears to be at the lowest point in the estuary, between Ilfracombe and Gower – contrary to earlier computations. The analytical solution clearly reveals the presence of progressive waves, and the important role of the far-field boundary condition in absorbing them. This appears to have been neglected in numerical models, which may explain the difference from the earlier results.

1. Introduction

Nearly a century ago, G.I.Taylor produced a simple analytical model of the tidal flow in the Severn estuary (Taylor, 1921). It appears in Lamb's account of the "canal theory of the tides" (Lamb, 1932, pp 267-278), of which it is a special case. The canal theory considers tidal flow as a longitudinal gravity wave in a channel. Following Lamb's notation, if the width of the channel is *b(x)* and its depth *h(x)*, both varying with position *x* along the channel, then the equation for the surface elevation $\eta(x,t)$ at time *t* is (Lamb, 1932, p.274):

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b}\frac{\partial}{\partial x}\left(hb\frac{\partial \eta}{\partial x}\right) \quad (1)$$

where *g* is the acceleration due to gravity. In an estuary, high tide is assumed to occur at the same time *t=0* everywhere, since the extent of the estuary, when measured in degrees of longitude, is small compared with the tidal cycle of approximately 180 degrees. A solution is therefore sought of the form:

$$\eta(x,t) = \eta_0(x)\cos(\omega t) \quad (2)$$

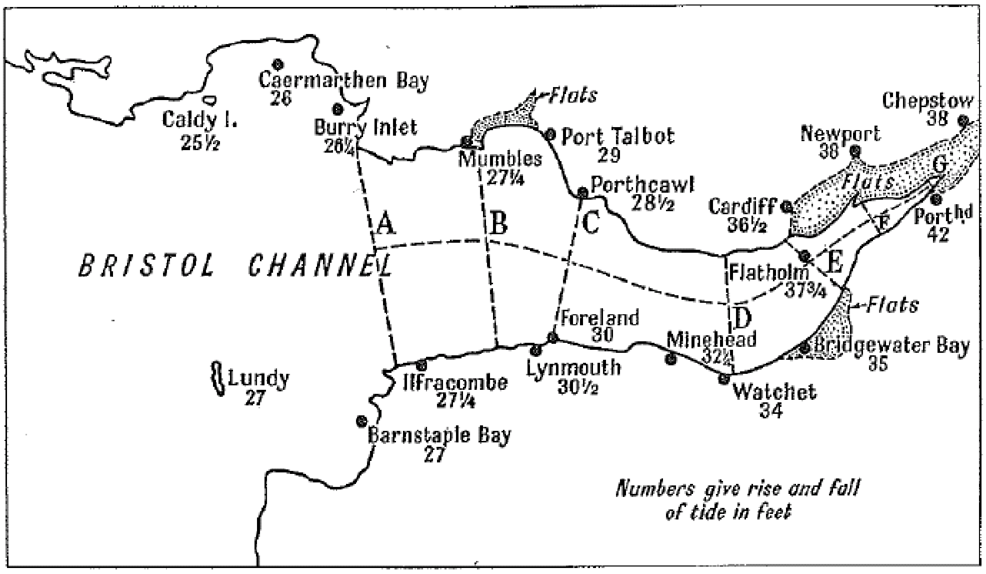
where $2\pi/\omega$ is the tidal period of approximately 12 hours (half a lunar day). Thus (1) becomes:

$$\frac{g}{b}\frac{d}{dx}\left(hb\frac{d\eta_0}{dx}\right)+\omega^2\eta_0=0 \qquad (3)$$

In the case of the Severn estuary, Taylor observed that the width *b(x)* and depth *h(x)* both increase approximately linearly with distance *x* downstream (referred to henceforth as "west") of the head of the estuary at Portishead, see Figure 1 below (originally Table 1 and Figure 1 in Taylor, 1921). He therefore took *x* = 0 at Portishead, and put:

$$b=\beta x \quad \text{and} \quad h=\gamma x \qquad (4)$$

where β,γ are constants. This reduces (3) to:



Taylor's Section	Taylor's measurements			Area upstream at mid-tide (sq. km)
	Distance from Portishead (km)	Mean depth at mid-tide (m)	Breadth at low tide (km)	
A	114.3	40.66	40.77	2962
B	92.10	31.58	37.06	2162
C	77.83	24.57	25.94	1577
D	46.33	19.81	22.24	882
E	28.72	13.99	12.97	499
G	0	11.89	3.25	136

Figure 1. Taylor's model of the Bristol Channel. His measurements have been converted to m and km. The upstream areas are from Admiralty Charts 1165, 1152, 1176 and 1166. Areas upstream of Sharpness are excluded.

$$\frac{d}{dx}\left(x^2\frac{d\eta_0}{dx}\right)+k\eta_0 x=0 \quad \text{with} \quad k=\omega^2/(\gamma g) \tag{5}$$

which can be solved exactly as a Bessel function:

$$\eta_0=\frac{KJ_1\{2\sqrt{kx}\}}{\sqrt{kx}} \tag{6}$$

where K is a constant. Taylor took γ= {25 fathoms}/{80 UK nautical miles} = 0.0003084 (β is immaterial) and the tidal period $2\pi/\omega$ as 12.4 hours, so that k = 0.00655 km^{-1}, and found (6) to be a good approximation to the observed variation of tidal range in the Severn estuary.

This paper extends Taylor's analysis to the case of a tidal power barrage in the estuary.

2. Tidal power - the need for progressive waves

Considered as a function of time, the horizontal velocity in a tidal wave (and indeed in a water wave generally) is 90 degrees out of phase with the surface slope $\partial\eta/\partial x$, since the latter is in phase with the horizontal acceleration. And the pressure variations are in phase with the surface elevation η. Thus for a standing-wave solution of the form (2), where the surface slope is in phase with the surface elevation, the velocity and pressure are 90 degrees out of phase. Therefore the power flux (= velocity × pressure) has a mean value of zero everywhere. This is of course to be expected, since the tidal energy is nowhere being dissipated in the estuary, only stored.

When we extract tidal power with a barrage, however, we require an equal mean power flux inwards at the mouth of the estuary. We thus reach the important conclusion that Taylor's solution (or any solution of the form (2)) is *inadmissible west of the barrage*, because it transmits no mean power. What is required west of the barrage is a *progressive wave*, in which there is a power flux, because the surface slope is 90 degrees out of phase with the surface elevation (and thus the velocity is in phase with the pressure). Rather than a solution of the form (2) we can seek a solution of the more general form

$$\eta(x,t)=\mathrm{Re}\{\eta_0(x)e^{i\omega t}\} \tag{7}$$

where $\eta_0(x)$ is now complex. This again leads to (5), which can be solved in the same way as:

$$\eta_0=\frac{K_1H_1\{2\sqrt{kx}\}+K_2\overline{H}_1\{2\sqrt{kx}\}}{\sqrt{kx}} \tag{8}$$

where H is a first-order Hankel function of the first kind, and we now have two constants K_1 and K_2. The first term is a progressive wave travelling east, and the second is a progressive wave travelling west. Far to the west, both resemble tidal waves in open water of the same depth (since $H(x) \sim -\{\cos(x+\pi/4) + i\sin(x+\pi/4)\}/\sqrt{x}$, for large x).

East of the barrage, no power is extracted (as before), so Taylor's solution remains admissible, but we are at liberty to change its amplitude and phase (i.e. the constant K in (6), and its phase).

3. An equivalent electric circuit

In his account of waves in channels, Lighthill (1978, p.104) introduces the electrical analogy of voltage with pressure, and electric current with volume flow rate. A reservoir of area S, in tidal waves sufficiently long for its surface to remain horizontal, is thus analogous to an electrical capacitance $S/\rho g$ (Lighthill, 1978, p.200 (3)), where ρ is the density of water. In our case, the reservoir area east of Taylor's sections A-G is given in Figure 1, and we can interpolate linearly between these values to give the area as a function $a(x)$ of the distance x west of Portishead. A strip of length dx thus has area $(da/dx)dx$, and its surface elevation, as a multiple of that at the barrage distance X to the west, is given by (6) as $[J_1\{2\sqrt{(kx)}\}/\sqrt{(kx)}]/[J_1\{2\sqrt{(kX)}\}/\sqrt{(kX)}]$. In addition, the area $a(0)$ east of Portishead has a surface elevation multiple $[J_1\{2\sqrt{(k0)}\}/\sqrt{(k0)}]/[J_1\{2\sqrt{(kX)}\}/\sqrt{(kX)}] = \sqrt{(kX)}/J_1\{2\sqrt{(kX)}\}$. We can thus obtain the capacitance C of the reservoir as:

$$C = \frac{\sqrt{kX}}{\rho g J_1\{2\sqrt{kX}\}}\left[\int_0^d \frac{J_1\{2\sqrt{kx}\}(da/dx)dx}{\sqrt{kx}} + a(0)\right] \qquad (9)$$

Following Lighthill's electrical analogy, we can write the impedance of the reservoir as $Z_1 = 1/(i\omega C)$.

West of the barrage, it is convenient to consider the water pressure variation (= ρg × level variation) as the sum of the pressure variation $Re\{Pe^{i\omega t}\}$ which would be seen in the absence of the barrage, and the additional pressure variation $Re\{P'e^{i\omega t}\}$ caused, immediately west of it, by the presence of the barrage. The additional pressure $Re\{P'e^{i\omega t}\}$ at the barrage produces a tidal wave which propagates out to sea - as far as the flow to the west of the barrage is concerned, the barrage is acting like a wave-maker. We require its wave-making impedance Z_2.

A unit wave propagating west is described by the second term in (8), with $K_2 = 1$. The water acceleration in this wave, in the direction of propagation, is minus the surface slope times g, whence we can obtain the water velocity in a westerly direction by integrating, as the real part of:

$$\frac{-g}{i\omega}\frac{d}{dx}\left(\frac{\overline{H_1}\{2\sqrt{kx}\}}{\sqrt{kx}}\right)e^{i\omega t} \qquad (10)$$

The volume flow rate in the direction of propagation is this velocity times *bh*, and the water pressure is $\rho g\eta$. We obtain the impedance Z_2 by dividing the latter by the former, which gives this impedance as:

$$\frac{-i\rho\omega\overline{H}_1\{2\sqrt{kx}\}}{bh\sqrt{kx}} \Bigg/ \frac{d}{dx}\left(\frac{\overline{H}_1\{2\sqrt{kx}\}}{\sqrt{kx}}\right) \qquad (11)$$

which we can consider as a resistance R in series with an inductance L, giving a combined impedance of $R + i\omega L$. For large x, the wave resembles a tidal wave in open water, for which the impedance is known to be purely a resistance of $\rho c/(bh)$ (Lighthill, 1978, p.104), where *c* is the open-water wave speed $\sqrt{(gh)}$. This gives a useful cross-check, when (11) is evaluated numerically.

In the absence of the barrier, the (complex) volume flow rate at the barrier location is P/Z_1, in an easterly direction. The additional wave-making volume flow immediately west of the barrier is P'/Z_2, in a westerly direction. Thus the total (complex) volume flow rate at this location, in an easterly direction, can be written:

$$\frac{P}{Z_1} - \frac{P'}{Z_2} \qquad (12)$$

If we write the total (complex) pressure at this location as $P'' = P + P'$, then (10) can be re-arranged to:

$$\frac{P\dfrac{Z_1 + Z_2}{Z_1} - P''}{Z_2} \qquad (13)$$

Thus the equivalent circuit of the channel west of the barrage is a voltage generator $P(Z_1 + Z_2)/Z_1$ with a source impedance of Z_2.

Considering the barrage itself and the reservoir east of it, the volume flow rate is:

$$\frac{P''}{R_B + Z_1} \qquad (14)$$

where R_B is flow-resistance of the turbines in the barrage itself, taken for simplicity as allowing flow in both directions (the most common arrangement, see Baker, 1991, p.31), with a constant resistance. Thus the equivalent circuit of the complete system is as shown in Figure 2 below.

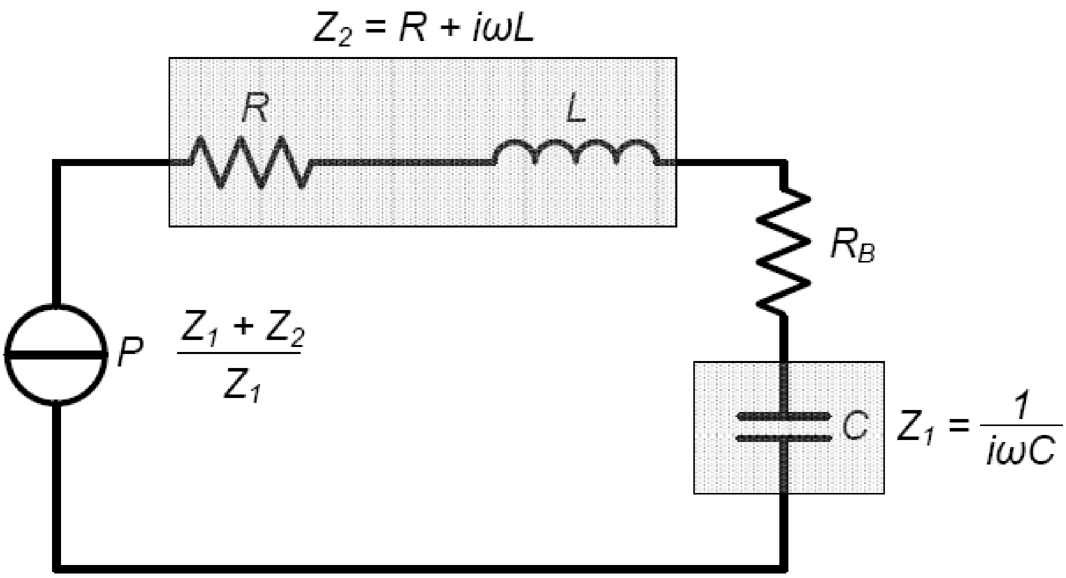


Figure 2. Equivalent electric circuit of barrage

When $R_B = 0$, it may be seen that the pressure at the barrage is its undisturbed value P, as it should be.

4. Similarity to wave power

At first sight it may seem curious that to provide the inward power flux needed to power the barrage, we introduce an additional tidal wave travelling in an *outward* direction. The reason is that from (8) Taylor's standing-wave solution (6) can be seen (by putting $K_1 = K_2 = K$ in (8) and noting that $H_1 + \overline{H_1} = J_1$) as the superposition of a tidal wave travelling east and an equal one travelling west. Our additional wave travelling west is cancelling part of his, giving a net inward wave.

This situation is familiar in wave power. See for example Rainey (2001) where the power flux in the circular far field around a wave power device is considered. The waves are considered in the standard manner of the water wave literature, as the sum of the incident wave, the diffracted wave produced by the presence of the stationary device, and the radiated wave produced by the motion of the device. The sum of the incident and diffracted waves has zero power flux, because the device is stationary, and therefore producing no power. It is analogous to Taylor's standing-wave solution for the Severn estuary, but more complicated because the far field is considered, where there is a net power input on the upwave side, which is cancelled by a net power output on the downwave side.

The radiated wave, although it is travelling outwards, upsets this power cancellation, and allows the wave power device to extract power, by moving in a suitable phase relationship to the incident waves.

5. Power available at various locations in the Severn estuary

From the equivalent circuit of Figure 2, the (complex) volume flow rate through the barrage is:

$$P\frac{Z_1+Z_2}{Z_1(Z_1+Z_2+R_B)} \quad (15)$$

and thus the power is:

$$\frac{1}{2}|P|^2\left|\frac{Z_1+Z_2}{Z_1(Z_1+Z_2+R_B)}\right|^2 R_B \quad (16)$$

This is readily calculated as a function of R_B, using the expression (9) and (11) above for Z_1 and Z_2. It is given in Figure 3 below for Taylor's sections A to E of Figure 1. The (complex) tidal pressure P in the absence of the barrage is taken as $4\rho g$ at Watchet, or 8m tidal range, which is the approximate root-mean-square value between the mean spring range of 10m, and the mean neap range of 5m, and thus gives the annual-average power. The values elsewhere are extrapolated from this 8m figure, using Taylor's formula (6).

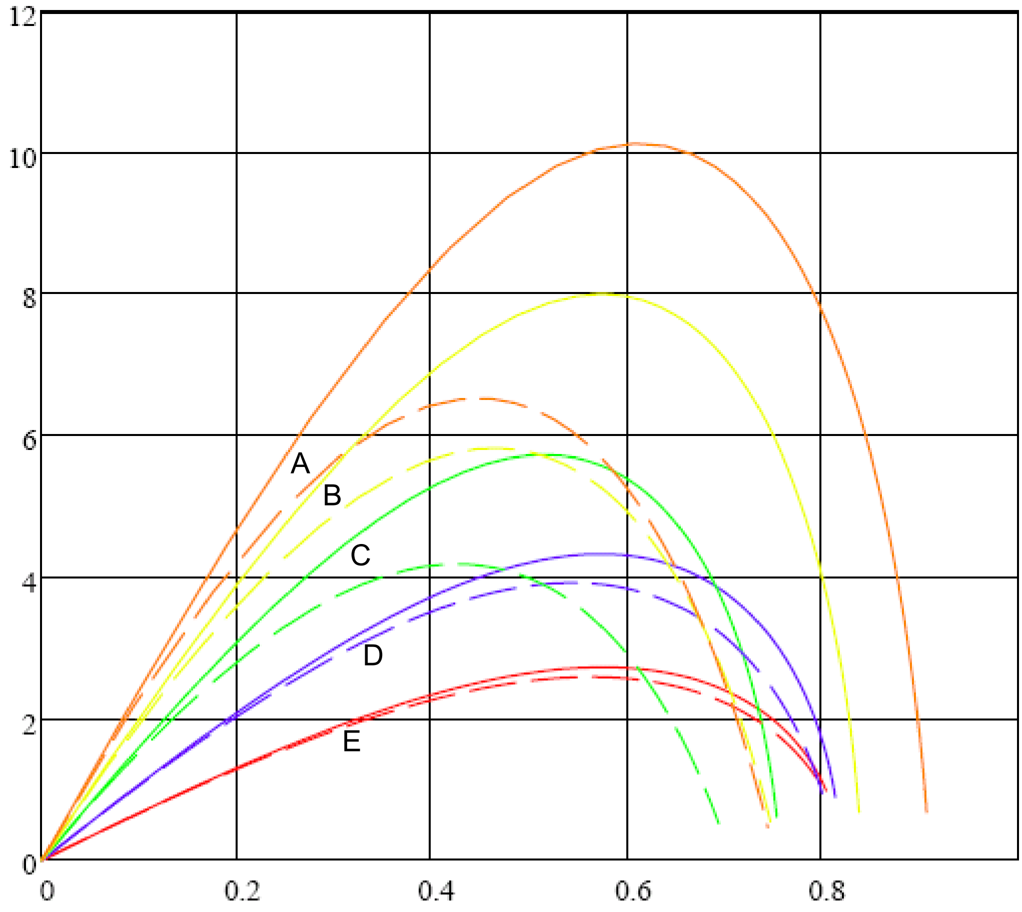


Figure 3. Power (GW) v. fraction of original tidal range across barrage, for barrages at locations A – E of Figure 1. Solid lines are with the outer estuary model (section 6) included, dashed lines are without it. In both cases the maximum power increases as the barrage is moved west.

Rather than plotting against R_B, Figure 3 is plotted against the pressure difference across the barrage, expressed as a fraction of the tidal pressure variation P in the absence of the barrage. Evidently the optimum value for this fraction is about 0.6, and the power increases steadily as the barrage is moved west. This is of course to be expected – as we move west, the reservoir area increases much more than the tidal range reduces, see Figure 1.

6. Effect of the shape of the estuary west of Taylor's model

Taylor observed that the shape of the Severn estuary changes abruptly west of his outer boundary (section A in Figure 2), and ceases to follow his formulae (4), even approximately. The width of the estuary approximately doubles immediately west of section A, and thereafter follows another of Taylor's linearly-tapering profiles, with both depth and width increasing approximately linearly with distance from a notional apex at Abergavenny, 100 km east of section A. The depth of 40m at section A gives a new value of γ^* = 40m/100km = 0.0004 for γ, and thus a new value k^* = 0.00505 km^{-1} for k. We now analyse the effect of this transition to a new profile.

The effect of the abrupt transition will be to reflect some of the wave travelling west considered in section 3, back up the channel. This reflection will be re-reflected from the barrage, and then again from the abrupt transition after section A, in an infinite sequence. We can sum all the waves travelling west into a single wave travelling west between the barrier and Taylor's section A, and likewise sum all the waves travelling east into a single wave travelling east in this region. We can write the (complex) volume flow rates in the direction of wave propagation as:

- V_O and V_B for the wave travelling west, at respectively the outer end of the region at section A, and at the barrier.
- V'_O and V'_B for the wave travelling east, at respectively the outer end of the region at section A, and at the barrier.

In the wave travelling west, the impedances at these locations are given by (11), we can write them as Z_O and Z_B. In the wave travelling east the impedances can be seen from (11) to be the complex conjugates of Z_O and Z_B – the Hankel function $\overline{H_1}$ from (8) becomes H_1, and the $-i$ from (10) becomes $+i$ because the acceleration in the direction of wave propagation is now plus the surface slope times g. In the region west of section A, we have only a wave travelling west, and the impedance is given by (11) with the new parameter k^* instead of k, and with x = 100 km. We can write this impedance as Z^*

We can now equate the sum of the pressures in the two waves immediately east of the transition at section A, with that in the single wave immediately west of it. The latter is obtained from the volume flow rate $V_O – V'_O$ in the westerly direction:

$$V_O Z_O + V'_O \overline{Z_O} = (V_O - V'_O) Z^* \qquad \text{i.e.} \qquad V'_O = \frac{Z^* - Z_O}{Z^* + \overline{Z_O}} V_O \qquad (17)$$

When $Z^* = Z_O$ there is no reflection from the outer boundary, and (17) accordingly predicts that $V'_O = 0$, as expected. In general we can define a reflection coefficient r:

$$r = \frac{Z^* - Z_O}{Z^* + \overline{\overline{Z_O}}} \qquad \text{so that} \qquad V'_O = rV_O \tag{18}$$

We now wish to deduce the impedance at the barrage, implied by (18). From the form of (8), the volume flow rates at the barrage and section A are related by:

$$V_O = V_B \mu e^{-i\omega T} \qquad \text{and} \qquad V'_B = V'_O \mu^{-1} e^{-i\omega T} \tag{19}$$

where T is the wave transit time between the barrage and section A (readily calculated from (8)), and μ is the (real) amplitude ratio of the two volume flow rates. Substituting (19) into (18), we obtain:

$$V'_B \mu e^{i\omega T} = rV_B \mu e^{-i\omega T} \qquad \text{or} \qquad V'_B = V_B r e^{-i2\omega T} \tag{20}$$

We can thus finally obtain the wave-making impedance at the barrage, as the sum of the pressures divided by the sum of the volume flow rates:

$$\frac{V_B Z_B + V_B r e^{-i2\omega T}\,\overline{Z_B}}{V_B - V_B r e^{-i2\omega T}} = \frac{Z_B + \overline{Z_B} r e^{-i2\omega T}}{1 - r e^{-i2\omega T}} \tag{21}$$

When $r = 0$, there is no reflection at the outer boundary and (21) then predicts that the wave-making impedance of the barrage is Z_B, as expected.

Using expression (11), we can calculate r from (18) and then from (21) the new wave-making barrage impedance Z_2 in Figure 2. The barrage powers can then be re-calculated from (16) – the results are shown in Figure 3. Evidently the changed shape of the estuary west of Taylor's original model increases the power somewhat, which is to be expected since the increased width of the estuary will lower Z_2 and thus from Figure 3 increase the power. The effect is more pronounced the closer the barrage is to this increased width. Thus the conclusion remains that the power increases steadily as the barrage is moved west – indeed it now increases more.

The changes in tidal range produced by the barrage are also noteworthy. They are readily calculated from Figure 3, using the full expression (21) for Z_2, and are shown in Figure 4 below, on the same horizontal axis as Figure 3. Taking into account the fact that the power peak in Figure 3 is further to the left for section C, the changes to the tidal range are very similar for all barrage locations. With barrages operated at maximum power, the tidal range is cut to 70% of its former value east of the barrage, and 90% of its former value immediately west of the barrage. A very simple view of the barrage is that (from (9) and (11)) Z_2 is small compared with Z_1. From Figure 2, the optimum power is when R_B has the same impedance as Z_1. This would reduce the tidal range east of the barrage by a factor √2, and leave the range immediately west of it unaffected, because Z_2 is small.

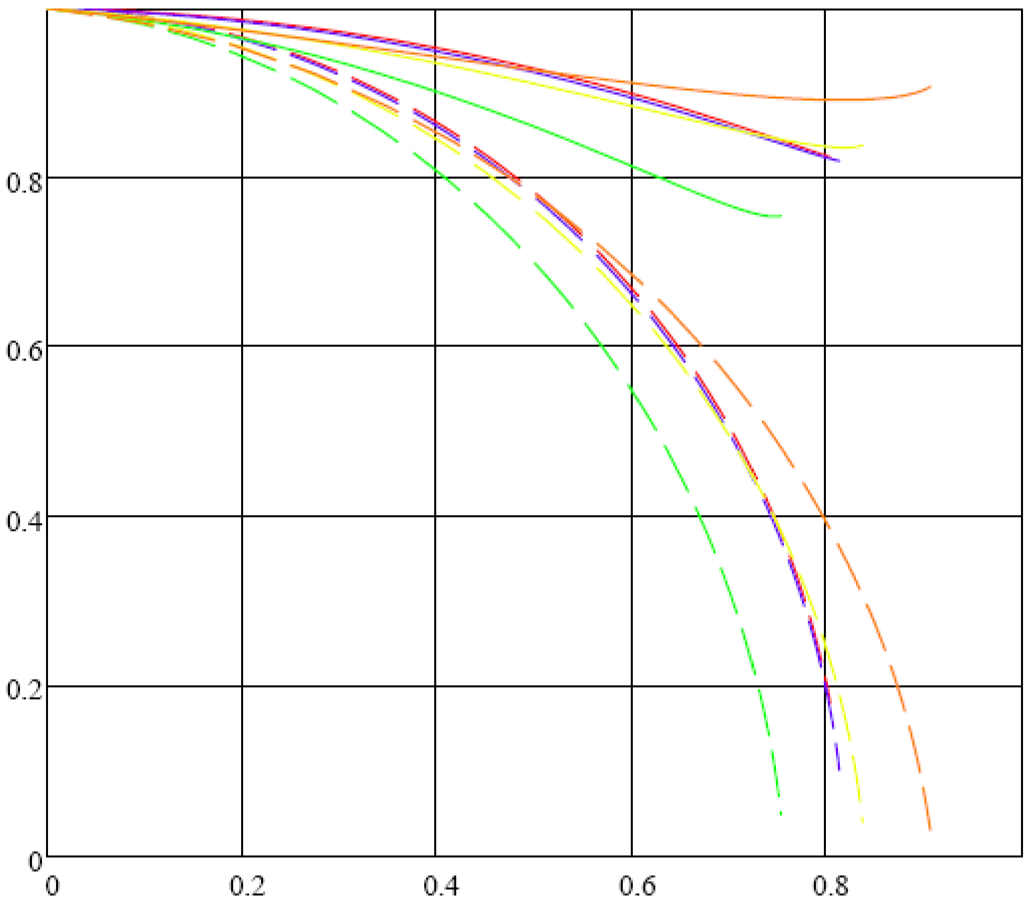


Figure 4. Fractional tidal change v. fraction of original tidal range, for barrages at locations A – E of Figure 1. Dashed lines are east of the barrage, solid lines are just west of it. The colour coding is the same as in Figure 3.

7. Previous computations

The question of the optimum position for a barrage in the Severn estuary, from the power point of view, was studied 30 years ago, see Bondi *et al.* (1981). The power was computed with various finite-difference numerical models, some of which extended out into the Irish Sea. They showed the average power rising strongly from 0.5 GW to 2.3 GW as the barrage was moved west from Taylor's section F to D (Bondi *et al.* 1981, Vol1 p.18). This is similar to the results in Figure 3, allowing for conversion losses. However, very little increase was found for positions further west. By Taylor's section C, the power was starting to decline, in marked contrast to the increase seen in Figure 3 – albeit that significant discrepancies were found between computer models (Bondi *et al.* 1981, vol 2 p.57).

We now explore a possible reason for this decline, which is that all the models simply held the tidal range fixed on the outer boundary, at the same value it would have were the barrage absent. This will produce a total reflection of the outgoing tidal wave – it is equivalent to setting Z^* in (18) equal to zero. This leads to:

$$V'_O = \frac{-Z_O}{\overline{Z_O}} V_O \qquad \text{i.e.} \qquad r = \frac{-Z_O}{\overline{Z_O}} = -e^{i2\varphi} \qquad (22)$$

where $\varphi = \arg(Z_O)$ is the phase advance of pressure over volume flow rate, in an outwards-propagating tidal wave, at the outer boundary. For an outer boundary at section A, for example, it can be calculated from (8) as 45.3 degrees. If we similarly write $\theta = \arg(Z_B)$ then $Z = \zeta e^{i\theta}$ where ζ is real and θ is the phase advance of pressure over volume flow rate, in an outwards-propagating tidal wave, at the outer barrage. For a barrage at section E, for example, it can be calculated from (8) as 68.9 degrees. The wave-making impedance Z_2 of the barrage (21) thus becomes:

$$\frac{\zeta e^{i\theta} - \zeta e^{-i\theta} e^{i2\varphi} e^{-i2\omega T}}{1 + e^{i2\varphi} e^{-i2\omega T}} = \frac{\zeta e^{i(\pi-\theta+2\varphi-2\omega T)} + \zeta e^{i\theta}}{e^{i(2\varphi-2\omega T)} + 1} \qquad (23)$$

Since $e^{i\chi} + e^{i\psi} = \{e^{i(\chi-\psi)/2} + e^{-i(\chi-\psi)/2}\}e^{i(\chi+\psi)/2} = 2\cos\{(\chi-\psi)/2\}e^{i(\chi+\psi)/2}$ this impedance can be written:

$$\frac{\zeta \cos\{\pi/2 + (\varphi - \theta - \omega T)\} e^{i(\pi/2+\varphi-\omega T)}}{\cos(\varphi - \omega T) e^{i(\varphi-\omega T)}} = i\zeta \frac{\sin(\omega T + \theta - \varphi)}{\cos(\omega T - \varphi)} \qquad (24)$$

Thus the wave-making impedance at the barrage is purely imaginary (i.e. reactive), as we would expect - the barrier can radiate no wave power, because the waves it sends west are perfectly reflected back by the outer boundary. Its amplitude is small if the far boundary is close to the barrier, because then θ and φ are nearly equal, and the phase delay ωT of in a tidal wave between the barrage to the outer boundary is then also small. Thus the change in the results will be small, because Z_2 is small anyway, as noted at the end of the previous Section.

However, when the barrier is a long way from the barrage, φ will be small because the tidal wave at the outer boundary will resemble an open-water wave. Thus when the phase delay ωT reaches 90 degrees, the denominator in (24) will drop to zero, and the wave-making impedance of the barrage will become very large. The power from the barrage will accordingly drop. This condition requires the transit time T of a tidal wave between the barrage and the outer boundary to be a quarter of the tidal period, or 12.4/4 = 3.1 hours. This is a resonant condition, with the natural sloshing period of the basin between the barrage and the outer boundary equal to the tidal period. With a mean tidal wave speed of 25m/s, say, it corresponds to a distance from the barrage to the outer boundary of 25×3600×4 = 360 km. This is comparable with the size of the larger models used by Bondi *et al.* (1981).

It is thus possible that the models used by Bondi *et al.* (1981) were giving spurious results due to internal resonances, caused by the incorrect outer boundary condition, in which the tidal range was held at the same value it would have were the barrage absent.

CALCULATION OF SEVERN BARRAGE POWERS - WITH HANKEL FUNCTIONS AND OUTER CHANNEL MODEL RCTR 30 Mar 09

First read in G.I.Tayor's (Scientific Papers Vol 2 pp.185 - 189) distances from Portishead, in UK nautical miles, and convert to m

$$x := \begin{pmatrix} 15.5 \\ 25 \\ 42 \\ 49.7 \\ 61.7 \end{pmatrix} \qquad x1 := 6080 \cdot 12 \cdot 0.0254 \cdot x \qquad x1 = \begin{pmatrix} 2.872 \times 10^{4} \\ 4.633 \times 10^{4} \\ 7.783 \times 10^{4} \\ 9.21 \times 10^{4} \\ 1.143 \times 10^{5} \end{pmatrix}$$

Input Taylor's breadth slope β, his depth slope γ and his tidal frequency ω, in rads/sec. Hence calculate his parameter k = ω^2/(γg) m^-1. Also input ρ in kg/m^3 and g in m/s^2

$$\beta := \frac{20}{50} \qquad \gamma := 0.0003084 \qquad \omega := \frac{2\cdot\pi}{12.4\cdot3600} \qquad i\omega := \frac{\pi\cdot 2i}{12.4\cdot3600} \qquad \omega = 1.408 \times 10^{-4}$$

$$i\omega = 1.408i \times 10^{-4} \qquad \rho := 1025 \qquad g := 9.81 \qquad k := \frac{\omega^2}{\gamma\cdot g} \qquad k = 6.548 \times 10^{-6}$$

Calculate tidal range at Taylor's positions, from his formula. Assume plus and minus 4m at second position x1 (Watchet).

$$j := 0..4$$

$$a_j := \frac{4\cdot\left[J1\left[2\cdot\left(k\cdot x1_j\right)^{0.5}\right]\cdot\left(k\cdot x1_1\right)^{0.5}\right]}{J1\left[2\cdot\left(k\cdot x1_1\right)^{0.5}\right]\cdot\left(k\cdot x1_j\right)^{0.5}} \qquad a = \begin{pmatrix} 4.248 \\ 4 \\ 3.58 \\ 3.399 \\ 3.129 \end{pmatrix} \qquad a_1 = 4$$

calculate capacitance "S/rhog" of reservoir above barrage assuming it is level. S is in m^2, and rho is in kg/m^3, so answer in m^3/Pa.

$$CL_j := \frac{\beta\cdot\left(x1_j\right)^2}{2\cdot\rho\cdot g} \qquad CL = \begin{pmatrix} 1.641 \times 10^{4} \\ 4.269 \times 10^{4} \\ 1.205 \times 10^{5} \\ 1.687 \times 10^{5} \\ 2.6 \times 10^{5} \end{pmatrix}$$

Calculate capacitance more accurately from actual reservoir areas A (in sq km - multiply by 1,000,000 for sq m), upstream of Taylor's sections, input as data. Include the area upstream of Portishead, so it becomes a 6-vector. Compare with Taylor's areas.

$$A := 1000000 \cdot \begin{pmatrix} 136 \\ 499 \\ 882 \\ 1577 \\ 2162 \\ 2962 \end{pmatrix} \qquad A = \begin{pmatrix} 1.36 \times 10^{8} \\ 4.99 \times 10^{8} \\ 8.82 \times 10^{8} \\ 1.577 \times 10^{9} \\ 2.162 \times 10^{9} \\ 2.962 \times 10^{9} \end{pmatrix} \qquad ATay_j := \frac{\beta \cdot \left(x1_j\right)^2}{2} \qquad ATay = \begin{pmatrix} 1.65 \times 10^{8} \\ 4.293 \times 10^{8} \\ 1.212 \times 10^{9} \\ 1.697 \times 10^{9} \\ 2.615 \times 10^{9} \end{pmatrix}$$

Define function to find the area at any position x downstream of Portishead, by linear interpolation. We need first to define a vector of positions corresponding to the 6 areas:

$$x0 := 0$$

$$xx := \mathrm{stack}(x0, x1)$$

$$AT(x) := \mathrm{linterp}(xx, A, x) \qquad AT(28720) = 4.989 \times 10^{8} \qquad AT(114000) = 2.95 \times 10^{9}$$

$$xx = \begin{pmatrix} 0 \\ 2.872 \times 10^{4} \\ 4.633 \times 10^{4} \\ 7.783 \times 10^{4} \\ 9.21 \times 10^{4} \\ 1.143 \times 10^{5} \end{pmatrix}$$

$$A_0 = 1.36 \times 10^{8}$$

$$C_j := \frac{\left(k \cdot x1_j\right)^{0.5}}{\rho \cdot g \cdot J1\left[2 \cdot \left(k \cdot x1_j\right)^{0.5}\right]} \cdot \left[\int_0^{x1_j} \frac{J1\left[2 \cdot (k \cdot y)^{0.5}\right] \cdot \left(\frac{d}{dy} AT(y)\right)}{(k \cdot y)^{0.5}} \, dy + A_0\right] \qquad C = \begin{pmatrix} 5.277 \times 10^{4} \\ 9.53 \times 10^{4} \\ 1.796 \times 10^{5} \\ 2.489 \times 10^{5} \\ 3.533 \times 10^{5} \end{pmatrix}$$

Calculate impedance Z1 = 1/(iωC) using these more accurate capactances. C is in m^3/Pa, so impedance will be in Pa/(m^3/s)

$$Z1_j := \frac{1}{i\omega \cdot C_j} \qquad Z1 = \begin{pmatrix} -0.135i \\ -0.075i \\ -0.04i \\ -0.029i \\ -0.02i \end{pmatrix}$$

Input Taylor's low-water depth + range/2, in feet, and convert to metres. Calculate local wave speed c in m/s.

$$h := \begin{pmatrix} 45.9 \\ 65 \\ 80.6 \\ 103.6 \\ 133.4 \end{pmatrix} \qquad h1 := h \cdot 12 \cdot 0.0254 \qquad c := (g \cdot h1)^{0.5} \qquad h1 = \begin{pmatrix} 13.99 \\ 19.812 \\ 24.567 \\ 31.577 \\ 40.66 \end{pmatrix} \qquad c = \begin{pmatrix} 11.715 \\ 13.941 \\ 15.524 \\ 17.6 \\ 19.972 \end{pmatrix}$$

Input Taylor's breadths in UK nautical miles, and convert to metres

$$b := \begin{pmatrix} 7 \\ 12 \\ 14 \\ 20 \\ 22 \end{pmatrix} \qquad b1 := 6080 \cdot 12 \cdot 0.0254 \cdot b \qquad b1 = \begin{pmatrix} 1.297 \times 10^{4} \\ 2.224 \times 10^{4} \\ 2.594 \times 10^{4} \\ 3.706 \times 10^{4} \\ 4.077 \times 10^{4} \end{pmatrix}$$

Calculate uniform-channel wavemaking resistances RU = ρc/(bh). ρ is in kg/m^3 and b is in m, so result will be in the mks unit Pa/(m^3/s)

$$RU_j := \frac{\rho \cdot c_j}{b1_j \cdot h1_j} \qquad RU = \begin{pmatrix} 0.066 \\ 0.032 \\ 0.025 \\ 0.015 \\ 0.012 \end{pmatrix}$$

OUTER CHANNEL MODEL. Beyond Taylor's section A we can define another Section AA, between the tips of Cornwall and Pembroke. In between we can define another Taylor model, with xo as the distance from its origin. This origin is 100km upstream of section A, and the depth ho = 40m and bo = 80km at section A increase to 60m and 120 km at CB, 50 km further out from section A (we also define a point AAA much further out, for checking purposes). We can thus obtain Taylor's k-parameter, written ko for the outer channel as:

$$xo := \begin{pmatrix} 100000 \\ 150000 \\ 100000000 \end{pmatrix} \qquad xo = \begin{pmatrix} 1 \times 10^{5} \\ 1.5 \times 10^{5} \\ 1 \times 10^{8} \end{pmatrix} \qquad \gamma o := \frac{40}{100000} \qquad ho := \gamma o \cdot xo \qquad ho = \begin{pmatrix} 40 \\ 60 \\ 4 \times 10^{4} \end{pmatrix}$$

$$ko := \frac{\omega^2}{\gamma o \cdot g} \qquad ko = 5.049 \times 10^{-6} \qquad \beta o := \frac{80}{100} \qquad bo := \beta o \cdot xo \qquad bo = \begin{pmatrix} 8 \times 10^{4} \\ 1.2 \times 10^{5} \\ 8 \times 10^{7} \end{pmatrix}$$

Now calculate imedances in an outward-propagating wave in this outer channel, using Taylor's solution with the multiplier e^iωt (omitted below) on the complex conjugate of the Hankel function - the real part is then an outward-propagating wave. First we write the (complex) pressure in Pa. We are taking the Hankel function as the height of the wave in metres (i.e.

Taylor's K=1m), and we thus obtain the pressure in Pa:

$jo := 0..2$

$$Po_{jo} := \frac{\rho \cdot g \cdot \overline{H1\left[1, 2 \cdot \left(ko \cdot xo_{jo}\right)^{0.5}\right]}}{\left(ko \cdot xo_{jo}\right)^{0.5}} \qquad Po = \begin{pmatrix} 7.722 \times 10^{3} + 6.578i \times 10^{3} \\ 6.699 \times 10^{3} + 3.002i \times 10^{3} \\ 9.506 + 52.414i \end{pmatrix}$$

Now define so(x) = d/dxo of the surface elevation:

$$so(xo) := \frac{d}{dxo} \frac{\overline{H1\left[1, 2 \cdot (ko \cdot xo)^{0.5}\right]}}{(ko \cdot xo)^{0.5}}$$

The outwards acceleration is - so.g. The outwards velocity is the outwards acceleration divided by (iω). Finally multiply the velocity by the area bh to get the volume flow rate in m^3/s

$$Vo_{jo} := \frac{bo_{jo} \cdot ho_{jo} \cdot (-g)}{i\omega} \cdot so\left(xo_{jo}\right) \qquad Vo = \begin{pmatrix} 2.235 \times 10^{6} - 4.74i \times 10^{5} \\ 2.548 \times 10^{6} - 9.758i \times 10^{5} \\ 5.609 \times 10^{7} + 2.596i \times 10^{8} \end{pmatrix}$$

Hence calculate the impedance as pressure/velocity.

$$Z21o_{jo} := \frac{Po_{jo}}{Vo_{jo}} \qquad Z21o_{jo} = \begin{pmatrix} 2.709 \times 10^{-3} + 3.518i \times 10^{-3} \\ 1.899 \times 10^{-3} + 1.906i \times 10^{-3} \\ 2.005 \times 10^{-7} + 6.693i \times 10^{-9} \end{pmatrix}$$

We can cross-check the impedance at the far boundary AAA against the impedance ρc/area of a uniform channel:

$$\frac{Z21o_{2}}{\frac{\rho \cdot \sqrt{g \cdot ho_{2}}}{bo_{2} \cdot ho_{2}}} = 0.999 + 0.033i$$

Input the impedance at the actual outer boundary AA (Pembroke-Cornwall):

$$ZOo := 1.0 \cdot Z21o_{1} \qquad ZOo = 1.899 \times 10^{-3} + 1.906i \times 10^{-3}$$

Hence calculate the reflection coefficient at the outer boundary AA

$$ro := \frac{ZOo - Z21o_1}{ZOo + \overline{Z21o_1}} \qquad ro = 0$$

Calculate the transit time To in seconds for wave propation from A to AA. The ratio of the Hankel functions at the two point gives the phase advance ph - minus this, divided by ω, gives the transit time in seconds.

$$pho := \arg\left(\frac{\overline{H1\left[1, 2\cdot\left(ko\cdot xo_1\right)^{0.5}\right]}}{\overline{H1\left[1, 2\cdot\left(ko\cdot xo_0\right)^{0.5}\right]}}\right) \qquad To := \frac{-pho}{\omega} \qquad pho = -0.284 \qquad To = 2.02\times 10^3$$

Hence calculate the wave-making impedance at A

$$ZO := \frac{Z21o_1 + \overline{Z21o_1}\cdot ro\cdot e^{-2i\cdot\omega\cdot To}}{1 - ro\cdot e^{-2i\cdot\omega\cdot To}}$$

$$ZO = 1.899\times 10^{-3} + 1.906i\times 10^{-3}$$

Now calculate the imedances of an outward-propagating wave between sections E and A, by the same method.

$$P_j := \frac{\rho\cdot g\cdot \overline{H1\left[1, 2\cdot\left(k\cdot x1_j\right)^{0.5}\right]}}{\left(k\cdot x1_j\right)^{0.5}} \qquad P = \begin{pmatrix} 9.139\times 10^3 + 2.1i\times 10^4 \\ 8.605\times 10^3 + 1.272i\times 10^4 \\ 7.701\times 10^3 + 6.483i\times 10^3 \\ 7.313\times 10^3 + 4.889i\times 10^3 \\ 6.732\times 10^3 + 3.089i\times 10^3 \end{pmatrix}$$

$$s(x) := \frac{d}{dx}\frac{\overline{H1\left[1, 2\cdot(k\cdot x)^{0.5}\right]}}{(k\cdot x)^{0.5}}$$

$$V_j := \frac{b1_j\cdot h1_j\cdot(-g)}{i\omega}\cdot s\left(x1_j\right) \qquad V = \begin{pmatrix} 9.093\times 10^5 - 3.888i\times 10^4 \\ 9.468\times 10^5 - 9.075i\times 10^4 \\ 5.683\times 10^5 - 1.223i\times 10^5 \\ 7.88\times 10^5 - 2.173i\times 10^5 \\ 7.757\times 10^5 - 2.923i\times 10^5 \end{pmatrix}$$

$$Z21_j := \frac{P_j}{V_j} \qquad Z21_j = \begin{pmatrix} 9.0467\times 10^{-3} + 0.02348i \\ 7.73003\times 10^{-3} + 0.01418i \\ 0.01061 + 0.01369i \\ 7.03495\times 10^{-3} + 8.14473i\times 10^{-3} \\ 6.28584\times 10^{-3} + 6.35101i\times 10^{-3} \end{pmatrix}$$

Now calculate the reflection coefficient at A:

$$r := \frac{ZO - Z21_4}{ZO + \overline{Z21_4}}$$

$$r = -0.186 - 0.644i$$

Calculate the transit time T in seconds for wave propation from E-A to A.

$$ph_j := \arg\left(\frac{\overline{H1\left[1, 2\cdot\left(k\cdot x1_4\right)^{0.5}\right]}}{H1\left[1, 2\cdot\left(k\cdot x1_j\right)^{0.5}\right]}\right) \qquad T_j := \frac{(-ph)_j}{\omega} \qquad ph = \begin{pmatrix} -0.73 \\ -0.546 \\ -0.27 \\ -0.159 \\ 0 \end{pmatrix} \qquad T = \begin{pmatrix} 5.187 \times 10^3 \\ 3.878 \times 10^3 \\ 1.915 \times 10^3 \\ 1.13 \times 10^3 \\ 0 \end{pmatrix}$$

Hence calculate the revised barrier impedance

$$Z2_j := \frac{Z21_j + \overline{Z21_j}\cdot r\cdot e^{-2i\cdot\omega\cdot T_j}}{1 - r\cdot e^{-2i\cdot\omega\cdot T_j}} \qquad Z2 = \begin{pmatrix} 1.797 \times 10^{-3} + 0.024i \\ 1.539 \times 10^{-3} + 0.013i \\ 2.403 \times 10^{-3} + 9.698i \times 10^{-3} \\ 1.755 \times 10^{-3} + 4.614i \times 10^{-3} \\ 1.899 \times 10^{-3} + 1.906i \times 10^{-3} \end{pmatrix}$$

Input a range of barrier resistances:

$$jj := 0..300$$

$$RB_{jj} := 0.002\cdot jj$$

Calculate the power for each. The answer will be in W, so divide by 1000000000 for GW.

$$PO_{j,jj} := \frac{1}{2000000000}\cdot\left(a_j\cdot\rho\cdot g\right)^2\cdot\left[\mathrm{Re}\left[\frac{Z1_j + Z2_j}{Z1_j\cdot\left(Z1_j + Z2_j + RB_{jj}\right)}\right]^2 + \mathrm{Im}\left[\frac{Z1_j + Z2_j}{Z1_j\cdot\left(Z1_j + Z2_j + RB_{jj}\right)}\right]^2\right]\cdot RB.$$

PO =

	0	1	2	3	4	5	6	7	8
0	0	0.101	0.201	0.301	0.4	0.498	0.595	0.691	0.785
1	0	0.29	0.578	0.861	1.137	1.406	1.666	1.915	2.153
2	0	0.816	1.594	2.317	2.973	3.554	4.057	4.482	4.832
3	0	1.406	2.725	3.913	4.943	5.805	6.5	7.038	7.437
4	0	2.367	4.48	6.249	7.641	8.671	9.382	9.829	10.068

Also calculate the pressure difference across the barrage, as a fraction F of the undisturbed tidal range:

$$F_{j,jj} := \left[\mathrm{Re}\left[\frac{(Z1_j + Z2_j)\cdot RB_{jj}}{Z1_j\cdot(Z1_j + Z2_j + RB_{jj})}\right]^2 + \mathrm{Im}\left[\frac{(Z1_j + Z2_j)\cdot RB_{jj}}{Z1_j\cdot(Z1_j + Z2_j + RB_{jj})}\right]^2\right]^{0.5}$$

F =

	0	1	2	3	4	5	6	7	8	9
0	0	0.015	0.03	0.044	0.059	0.074	0.088	0.103	0.117	0.132
1	0	0.027	0.053	0.08	0.106	0.132	0.157	0.182	0.206	0.23
2	0	0.05	0.099	0.146	0.192	0.234	0.274	0.311	0.345	0.377
3	0	0.069	0.137	0.2	0.26	0.315	0.365	0.411	0.451	0.488
4	0	0.098	0.19	0.275	0.351	0.418	0.477	0.527	0.57	0.607

Repeat power and pressure difference calculation without outer channel model, i.e. with Z21 instead of Z2

$$PO1_{j,jj} := \frac{1}{2000000000}\cdot(a_j\cdot\rho\cdot g)^2\cdot\left[\mathrm{Re}\left[\frac{Z1_j + Z21_j}{Z1_j\cdot(Z1_j + Z21_j + RB_{jj})}\right]^2 + \mathrm{Im}\left[\frac{Z1_j + Z21_j}{Z1_j\cdot(Z1_j + Z21_j + RB_{jj})}\right]^2\right]\cdot$$

PO1 =

	0	1	2	3	4	5	6	7	8	9
0	0	0.1	0.2	0.299	0.396	0.492	0.587	0.68	0.772	0.861
1	0	0.288	0.57	0.844	1.108	1.362	1.604	1.834	2.05	2.253
2	0	0.782	1.467	2.055	2.55	2.959	3.291	3.555	3.76	3.916
3	0	1.341	2.482	3.417	4.158	4.725	5.145	5.443	5.641	5.761
4	0	2.172	3.797	4.94	5.697	6.163	6.419	6.528	6.534	6.472

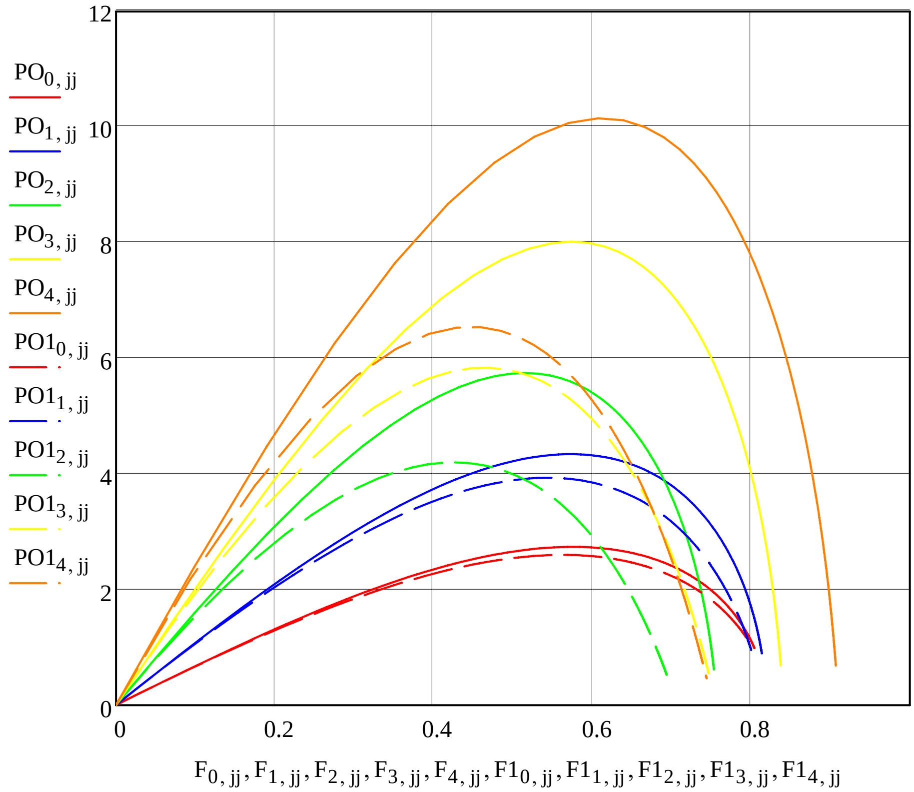
Also calculate the pressure difference across the barrage, as a fraction F of the undisturbed tidal range:

$$F1_{j,jj} := \left[\mathrm{Re}\left[\frac{(Z1_j + Z21_j)\cdot RB_{jj}}{Z1_j\cdot(Z1_j + Z21_j + RB_{jj})}\right]^2 + \mathrm{Im}\left[\frac{(Z1_j + Z21_j)\cdot RB_{jj}}{Z1_j\cdot(Z1_j + Z21_j + RB_{jj})}\right]^2\right]^{0.5}$$

F1 =

	0	1	2	3	4	5	6	7	8	9
0	0	0.015	0.03	0.044	0.059	0.073	0.088	0.102	0.116	0.13
1	0	0.027	0.053	0.079	0.105	0.13	0.154	0.178	0.201	0.224
2	0	0.049	0.095	0.138	0.177	0.214	0.247	0.277	0.305	0.33
3	0	0.068	0.13	0.187	0.239	0.284	0.325	0.361	0.393	0.421
4	0	0.094	0.175	0.245	0.303	0.353	0.394	0.43	0.46	0.485

Plot power against fraction F



Also calculate the fractional increases FU and FD in pressures upstream and downstream

$$FU_{j,jj} := \left[\mathrm{Re}\left[\frac{\left(Z1_j + Z2_j\right) \cdot Z1_j}{Z1_j \cdot \left(Z1_j + Z2_j + RB_{jj}\right)} \right]^2 + \mathrm{Im}\left[\frac{\left(Z1_j + Z2_j\right) \cdot Z1_j}{Z1_j \cdot \left(Z1_j + Z2_j + RB_{jj}\right)} \right]^2 \right]^{0.5}$$

$$FD_{j,jj} := \left[\mathrm{Re}\left[\frac{\left(Z1_j + Z2_j\right) \cdot \left(Z1_j + RB_{jj}\right)}{Z1_j \cdot \left(Z1_j + Z2_j + RB_{jj}\right)} \right]^2 + \mathrm{Im}\left[\frac{\left(Z1_j + Z2_j\right) \cdot \left(Z1_j + RB_{jj}\right)}{Z1_j \cdot \left(Z1_j + Z2_j + RB_{jj}\right)} \right]^2 \right]^{0.5}$$

FU =

	0	1	2	3	4	5	6	7	8	9
0	1	1	0.999	0.998	0.996	0.994	0.992	0.99	0.987	0.984
1	1	0.999	0.996	0.993	0.988	0.983	0.977	0.969	0.961	0.953
2	1	0.993	0.981	0.966	0.947	0.926	0.904	0.879	0.854	0.828
3	1	0.991	0.975	0.954	0.929	0.9	0.869	0.838	0.805	0.773
4	1	0.983	0.956	0.922	0.883	0.842	0.799	0.757	0.717	0.679

FD =

	0	1	2	3	4	5	6	7	8	9
0	1	1	0.999	0.999	0.998	0.997	0.996	0.995	0.994	0.993
1	1	0.999	0.998	0.996	0.994	0.992	0.989	0.986	0.983	0.98
2	1	0.994	0.986	0.977	0.967	0.956	0.944	0.933	0.921	0.91
3	1	0.993	0.984	0.975	0.964	0.954	0.943	0.933	0.923	0.914
4	1	0.988	0.975	0.963	0.951	0.94	0.931	0.923	0.916	0.911

4	1	0.988	0.976	0.965	0.951	0.94	0.931	0.928	0.916	0.911

Plot FU and FD against fraction F